THE GAUSS-GREEN THEOREM

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1. Introduction. All conventions of our papers on *Surface area*(1) are again in force. The positive integers $m \leqq n$ which were fixed throughout SA II are now so specialized that $m = n-1$, $n \geqq 2$. The corresponding(2) function Φ is an $(n-1)$-dimensional measure over Euclidean n-space, which reduces to Carathéodory linear measure if $n=2$.

The starting point of the present article is the definition of the *exterior normal* of a subset A of n-space at a point x in n-space. A glance at 3.1–3.4 below will convince the reader that the existence or nonexistence of this vector is a local, geometric property of the set A at the point x. None but the most elementary topology enters into this definition in which the boundary of A is never even mentioned. As will be reaffirmed in 3.6, $\nu(A, x)$ is the exterior normal of A at x whenever this exists, otherwise $\nu(A, x) = \theta$, the zero vector.

With these definitions of surface measure and exterior normal at hand, we are led to investigate the validity of the Gauss-Green formula

$$\int_A D_j f(x) dx = \int f(x) \nu_j(A, x) d\Phi x. \tag{1}$$

Here A is an open subset of E_n, j is a positive integer between 1 and n, $\nu_j(A, x)$ is the jth component of $\nu(A, x)$, and $D_j f$ is the partial derivative of f in the direction of the jth unit vector. In this connection we shall always make the assumptions that both integrals are finite, that the boundary of A has finite Φ measure, and that f is absolutely continuous within the closure of A along almost all lines in the direction of the jth unit vector(3).

It is seen from Theorem 6.4 that (1) is true if the above conditions are satisfied and if, in addition, the boundary of A has a certain(4) type of regularity. However we are able to prove much more in the special case $n=2$. In fact it is shown in §7 that the boundary of every open set in the plane has the required regularity if it is of finite Carathéodory linear measure. Thus (1) holds in the plane under the conditions of the preceding paragraph. The

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(1) See §2 of *Surface area*. I, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 420–437, and §§2, 3 of *Surface area*. II, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 438–456. We hereafter refer to these papers as SA I and SA II.

(2) See SA II, 2.1. The special case $n=3$ is also treated in SA I, 3.1–3.3.

(3) See 6.1 and 6.3.

(4) See 3.8–3.14 and the condition (II) of 6.4.

question whether this much is true in higher dimensions is left unanswered.

The above mentioned regularity of the boundary of A is called Φ *restrictedness*. Its local character is clear from Definition 3.12. Anthony P. Morse and John F. Randolph have recently(5) introduced and investigated this concept in the case of plane sets. The author wishes to express his sincere thanks to them for the opportunity to read their paper in manuscript. He has freely used their methods and results. Their paper and Randolph's thesis(6) have been his main source of interest in the subject matter of the present article.

§2 contains a theorem about the transformation of integrals. A generalization of the *strong* form of Cauchy's Theorem, concerning the integral of an analytic function *"around"* an open set, is proved in §8.

2. Transformation of integrals.

2.1 THEOREM. *If*

(I) *ϕ is such a measure over A that A is expressible as a countable sum of ϕ measurable sets of finite ϕ measure;*

(II) *ψ is such a measure over the metric space B that closed subsets of B are ψ measurable and every ψ measurable set is contained in a Borel set of equal ψ measure;*

(III) *g is such a function on A to B that*

$$\underset{x}{E}\,[g(x) \in Y]$$

is ϕ measurable for every closed set $Y \subset B$;

(IV) *u is such a ϕ measurable function that $0 \leqq u(x) < \infty$ for ϕ almost all x in A and*

$$\int N(g, X, y)d\psi y = \int_X u(x)d\phi x$$

for every ϕ measurable set $X \subset A$;

then

$$\int f(y)N(g, X, y)d\psi y = \int_X f[g(x)]u(x)d\phi x$$

whenever X is a ϕ measurable subset of A and f is such a ψ measurable function that $-\infty \leqq f(y) \leqq \infty$ for ψ almost all y in B.

Proof. We fix a ϕ measurable set $C \subset A$ and let

(5) A. P. Morse and John F. Randolph, *The ϕ rectifiable subsets of the plane*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305. We hereafter refer to this paper as RM. The bibliography of this paper gives references to previous articles on this subject, in particular the work of A. S. Besicovitch on plane sets of finite Carathéodory linear measure.

(6) John F. Randolph, *Carathéodory linear measure and a generalization of the Gauss-Green Lemma*, Trans. Amer. Math. Soc. vol. 38 (1935) pp. 531–548. The first attempt in this direction was made by J. Schauder, *The theory of surface measure*, Fund. Math. vol. 8 (1926) pp. 1–48.

$$P = C \underset{x}{E} [0 < u(x) < \infty].$$

Whenever r and f are functions on A and B respectively we denote

$$\alpha(r) = \int_P r(x)u(x)d\phi x,$$

$$\beta(f) = \int f(y)N(g, P, y)d\psi y.$$

Let F be the set of all such ψ measurable functions f that $-\infty \leqq f(y) \leqq \infty$ for ψ almost all y in B, and take

$$H = F \underset{f}{E} [f(y) = 0 \text{ for } y \in B - g^*(P)].$$

We further define for each $f \in F$ the function $\bar{f}$ on A by the relation

$$\bar{f}(x) = f[g(x)] \quad \text{for} \quad x \in A,$$

and divide the remainder of the proof into four parts.

Part 1. $g^*(P)$ is expressible as a countable sum of ψ measurable sets of finite ψ measure.

Proof. Use (I) to select ϕ measurable sets $A_1, A_2, A_3, \cdots$ such that

$$A = \sum_{j=1}^{\infty} A_j \text{ and } \phi(A_j) < \infty \qquad \text{for } j = 1, 2, 3, \cdots.$$

Letting

$$P_k = P \underset{x}{E} [u(x) < k] \qquad \text{for } k = 1, 2, 3, \cdots,$$

we see that

$$g^*(P) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} g^*(A_jP_k)$$

and

$$g^*(A_jP_k) = \underset{y}{E} [N(g, P_jA_k, y) \geqq 1],$$

$$\infty > k\phi(A_jP_k) \geqq \int_{A_jP_k} u(x)d\phi x = \int N(g, P_jA_k, y)d\psi y \geqq \psi[g^*(A_jP_k)]$$

for every pair of positive integers j and k.

Part 2. The set

$$P \underset{x}{E} [g(x) \in Y]$$

is ϕ measurable for every ψ measurable subset Y of $g^*(P)$.

Proof. Let G be the family of all those subsets Y of B for which the set

$$P \underset{x}{E} [g(x) \in Y]$$

is ϕ measurable.

Evidently G is closed to countable addition and to complementation.

Hence (III) implies that every Borel set of B is a member of G. Next we infer from Part 1 and (II) that every ψ measurable subset of $g^*(P)$ is expressible[7] as an F_σ plus a set of ψ measure zero. We shall complete the proof by showing that every subset of $g^*(P)$ of ψ measure zero is a member of G.

Suppose $Y \subset g^*(P)$, $\psi(Y)=0$ and

$$X = P \underset{x}{E} [g(x) \in Y].$$

Use (II) to select a Borel set Y_1 of B for which $Y \subset Y_1$ and $\psi(Y_1)=0$. From the preceding paragraph we see that $Y_1 \in G$. Let

$$X_1 = P\underset{x}{E}[g(x) \in Y_1]$$

and use (IV) together with the relation $\psi[g^*(X_1)] \leqq \psi(Y_1)=0$ to infer

$$0 = \int N(g, X_1, y)d\psi y = \int_{X_1} u(x)d\phi x.$$

But $X_1 \subset P$ so that $\phi(X_1)=0$. Since $X \subset X_1$, we conclude $\phi(X)=0$ and $Y \in G$.

Part 3. $f \in H$ implies $\beta(f)=\alpha(\bar{f})$.

Proof. In case f is the characteristic function of a ψ measurable set $Y \subset g^*(P)$, we let

$$X = \underset{x}{E} [g(x) \in Y],$$

note that $\bar{f}$ is the characteristic function of X, and use Part 2 and (IV) to infer

$$\alpha(\bar{f}) = \int_{PX} u(x)d\phi x = \int N(g, PX, y)d\psi y = \int_Y N(g, P, y)d\psi y = \beta(f).$$

But the functions α, β and $^-$ are additive, homogeneous, and continuous with respect to monotone convergence of nonnegative members of their domains. Hence

$$\beta(f) = \alpha(\bar{f}) \quad \text{for every nonnegative} \quad f \in H.$$

Now suppose f is an arbitrary member of H. Let f_1 and f_2 be such nonnegative functions in H that $f=f_1-f_2$. Then $\bar{f}_1$ and $\bar{f}_2$ are nonnegative functions for which $\bar{f}=\bar{f}_1-\bar{f}_2$, and we infer from the definition of the Lebesgue integral that

$$\beta(f) = \beta(f_2) - \beta(f_1) = \alpha(\bar{f}_2) - \alpha(\bar{f}_1) = \alpha(\bar{f}).$$

Part 4. If $f \in F$, then $\int f(y)N(g, C, y)d\psi y = \int_C \bar{f}(x)u(x)d\phi x$.

Proof. Select $h \in H$ so that

$$h(y) = f(y) \quad \text{for} \quad y \in g^*(P), \qquad h(y) = 0 \quad \text{for} \quad y \in B - g^*(P).$$

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(7) See Theorem 3.13 of RM.

Check the relations:

$$\text{(1)} \qquad \beta(h) = \alpha(\bar{h});$$

$$\bar{h}(x) = \bar{f}(x) \quad \text{for} \quad x \in P,$$

$$\text{(2)} \qquad \alpha(\bar{h}) = \alpha(\bar{f}) = \int_C \bar{f}(x)u(x)d\phi x;$$

$$0 = \int_{C-P} u(x)d\phi x = \int N(g, C - P, y)d\psi y,$$

$N(g, C, y) = N(g, P, y)$ for ψ almost all y in B,

$$\text{(3)} \qquad \int f(y)N(g, C, y)d\psi y = \beta(f) = \beta(h).$$

Combine (3), (1), and (2).

3. Definitions.

3.1 *Notation.*

$$x \cdot y = \sum_{j=1}^{n} x_j y_j \quad \text{for} \quad x \in E_n,\ y \in E_n,$$

$$K_x^r = E_n \mathop{E}_z [\,|z - x| < r] \quad \text{for} \quad x \in E_n,\ r > 0.$$

3.2 Definition. We say *u points into S at x* if and only if $S \subset E_n$, $x \in E_n$, $u \in E_n$, $|u| = 1$, and

$$\lim_{r \to 0+} \frac{|H_r - S|}{|H_r|} = 0,$$

where, for $r > 0$, H_r is the hemisphere

$$K_x^r \mathop{E}_z [(z - x) \cdot u > 0].$$

3.3 Definition. We call u an *exterior normal of S at x* if and only if u points into $(E_n - S)$ at x and $(-u)$ points into S at x.

3.4 Theorem. *If u and v are exterior normals of S at x, then $u = v$.*

Proof. For $w \in E_n$ and $r > 0$ we write

$$H_r^w = K_x^r \mathop{E}_z [(z - x) \cdot w > 0].$$

First we prove

$$\text{(1)} \qquad H_3^u H_3^{-v} = 0.$$

In fact the denial of (1) implies

$$|H_r^u H_r^{-v}| = r^n \, |H_1^u H_1^{-v}| > 0 \quad \text{for} \quad r > 0,$$

and since

$$\lim_{r \to 0+} \frac{|H_r^u S|}{|H_r^u|} = 0, \qquad \lim_{r \to 0+} \frac{|H_r^{-v} - S|}{|H_r^{-v}|} = 0,$$

it is obvious that

$$\lim_{r \to 0+} \frac{|H_r^u H_r^{-v} S|}{|H_r^u H_r^{-v}|} = 0, \qquad \lim_{r \to 0+} \frac{|H_r^{-v} H_r^u - S|}{|H_r^{-v} H_r^u|} = 0.$$

Adding we obtain

$$\lim_{r \to 0+} \frac{|H_r^u H_r^{-v}|}{|H_r^u H_r^{-v}|} = 0$$

which is false. Thus (1) is proved.

Now let $z = x + u - v$. Then the assumption $u \cdot v < 1$ implies $(z-x) \cdot u = (u-v) \cdot u = 1 - v \cdot u > 0$, $z \in H_3^u$; thus (1) yields the inequality $0 \geq (z-x) \cdot (-v) = (u-v) \cdot (-v) = 1 - u \cdot v > 0$, which is false. Hence $u \cdot v \geq 1$ and $u = v$, because $|u| = |v| = 1$.

3.5 *Notation.* We fix the points θ and i so that

$$\theta = (0, \cdots, 0) \in E_n, \qquad i = (0, \cdots, 0, 1) \in E_n.$$

3.6 *Notation.* For $S \subset E_n$ and $x \in E_n$ we define $\nu(S, x)$ as follows:

If S has a unique exterior normal u at x, then $\nu(S, x) = u$; otherwise $\nu(S, x) = \theta$.

3.7 *Notation*[8].

$$\{x\} = \underset{y}{E}\,[y = x].$$

3.8 DEFINITION. If $S \subset E_n$ and $x \in E_n$, then sgn (S, x) is the closure of the set

$$\sum_{y \in T} \left\{ \frac{y - x}{|y - x|} \right\}$$

where $T = S - \{x\}$.

3.9 DEFINITION. If ϕ is a measure over E_n, $S \subset E_n$ and $x \in E_n$, then we define the *upper* and *lower ϕ density of S at x* by the relations

$$\overset{\triangle}{D}_\phi(S, x) = \limsup_{r \to 0+} \frac{\phi(SK_x^r)}{\gamma(K_x^r)}, \qquad \overset{\triangledown}{D}_\phi(S, x) = \liminf_{r \to 0+} \frac{\phi(SK_x^r)}{\gamma(K_x^r)}.$$

From Definition 2.1 of SA II it is seen that $\gamma(K_x^r)$ equals the $(n-1)$-dimensional Lebesgue measure of an $(n-1)$-dimensional sphere of radius r.

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[8] Throughout this paper we use braces only in this sense.

3.10 DEFINITION. If ϕ is a measure over E_n, $S\subset E_n$ and $x\in E_n$, then

$$\operatorname{dir}_\phi (S, x) = \prod_{\beta\in F} \operatorname{sgn} (S - \beta, x)$$

where

$$F = \mathop{E}_{\beta} [\overset{\triangle}{D}_\phi (\beta, x) = 0].$$

3.11 DEFINITION. We say S is ϕ *restricted at* x if and only if

$$\overset{\triangle}{D}_\phi (S, x) > 0$$

and there is a point $z\in E_n$ for which $|z| = 1$ and

$$(\{z\} + \{-z\}) \operatorname{dir}_\phi (S, x) = 0.$$

3.12 DEFINITION. We say z is ϕ *perpendicular to* S *at* x if and only if

$$\overset{\triangle}{D}_\phi (S, x) > 0; \qquad z \in E_n, \quad |z| = 1;$$
$$\operatorname{dir}_\phi (S, x) \subset \mathop{E}_{y} [y\cdot z = 0].$$

3.13 DEFINITION. $x' = (x_1, x_2, \cdots, x_{n-1}) \in E_{n-1}$ for $x\in E_n$,

$$S' = \sum_{x\in S} \{x'\} \quad \text{for} \quad S \subset E_n;$$

$$(y \circ t) = (y_1, y_2, \cdots, y_{n-1}, t) \in E_n \quad \text{for} \quad y \in E_{n-1}, \qquad t \in E_1.$$

For $A \subset E_n$ and $k = 1, 2, 3, \cdots$ we let

$$\overset{+}{\Lambda}_k(A) = E_n \mathop{E}_{x} [0 < t - x_n < k^{-1} \text{ implies } (x' \circ t) \in A],$$

$$\overset{-}{\Lambda}_k(A) = E_n \mathop{E}_{x} [0 < x_n - t < k^{-1} \text{ implies } (x' \circ t) \in A],$$

$$\overset{+}{\Lambda}(A) = \sum_{k=1}^{\infty} \overset{+}{\Lambda}_k(A), \qquad \overset{-}{\Lambda}(A) = \sum_{k=1}^{\infty} \overset{-}{\Lambda}_k(A).$$

3.14 DEFINITION. For each function g on E_{n-1} to E_n we define the function τg on the set

$$\mathop{E}_{y} [Jg(y) > 0]$$

to E_n as follows:

Suppose $Jg(y) > 0$ and L is the approximate differential of g at y. For each integer k between 1 and k, strike out the kth row of L to obtain the minor M_k. Let $\tau g(y)$ be the point of E_n whose kth coordinate is

$$[\tau g(y)]_k = (-1)^{n+k}(\det M_k)/Jg(y).$$

It is well known that $\tau g(y)$ is perpendicular to the columns $L^1, L^2, \cdots, L^{n-1}$ of L.

3.15 REMARK. *If $A \subset E_n$ with $\Phi(A) < \infty$, then*

$$D_\Phi^{\triangle}(A, x) \leqq 1 \quad \textit{for } \Phi \textit{ almost all } x \textit{ in } A.$$

If furthermore B is a Borel set contained in A, then

$$D_\Phi^{\triangle}(A, x) = D_\Phi^{\triangle}(B, x) \quad \textit{and} \quad D_\Phi^{\triangledown}(A, x) = D_\Phi^{\triangledown}(B, x)$$

for Φ almost all x in B, and

$$\text{dir}_\Phi(B, x) = \text{dir}_\Phi(A, x)$$

for Φ almost all x in B.

These statements are the analogues of 5.16, 5.15 and 6.11 of RM. Their proofs are likewise analogous: Φ replaces L, and $\gamma(K_x^r)$ replaces $2r$.

Two sets, one of which is a measure hull of the other, have everywhere the same densities and the same dir. Hence the assumption that B is a Borel set can be omitted.

4. The exterior normal.

4.1 LEMMA. *If S is an open*(9) *subset of E_n, $|S| < \infty$, $|S'| < \infty$, and $t<1$, then there is a number u such that $u<1$ and*

$$|X| \geqq u|S| \quad \textit{implies} \quad |X'| \geqq t|S'|$$

whenever $X \subset S$ and X is of class F_σ.

Proof. We assume $|S'|>0$ and define

$$\begin{aligned} h(y) &= \left| E_1 \mathop{E}_v [(y \circ v) \in S] \right| \quad \text{for} \quad y \in S', \\ A(v) &= S' \mathop{E}_y [h(y) > v] \quad \text{for} \quad -\infty \leqq v \leqq \infty, \\ w &= \sup \mathop{E}_v [|A(v)| \geqq t|S'|], \\ B &= S' \mathop{E}_y [h(y) \geqq w]. \end{aligned}$$

Clearly $A(w) \subset B$. Furthermore $|S'| < \infty$ implies

$$|A(w)| = \sup_{v>w} |A(v)| \leqq t|S'| \leqq \inf_{v<w} |A(v)| = |B|.$$

Hence we can and do select a Lebesgue measurable set T for which

$$A(w) \subset T \subset B \quad \text{and} \quad |T| = t|S'|.$$

Next we take

$$u = \frac{\int_T h(y)dy}{|S|}.$$

(9) It is sufficient to assume that $h(y)>0$ for almost all y in S', where h is defined as in the proof. This condition is really necessary in case S is measurable. In case S is nonmeasurable the lemma is obvious.

In order to prove that

$$u < 1, \tag{1}$$

we suppose $u \geqq 1$. Remember that S is open and $h(y) > 0$ for $y \in S'$. Check the relations:

$$\int_T h(y)dy \geqq |S| = \int_{S'} h(y)dy, \quad T \subset S'; \qquad \int_{S'-T} h(y)dy = 0;$$

$$|S' - T| = 0; \qquad t|S'| = |T| = |S'|, \quad 0 < |S'| < \infty; \qquad t = 1.$$

Since $t < 1$, we have verified (1).

Now choose $X \subset S$ so that X is a set of class F_σ for which $|X| \geqq u|S|$ and suppose $|X'| < t|S'|$. Then

$$\int_{X'-T} h(y)dy + \int_{X'T} h(y)dy = \int_{X'} h(y)dy = |X| \geqq u|S|$$

$$= \int_T h(y)dy = \int_{T-X'} h(y)dy + \int_{TX'} h(y)dy,$$

$$w|X' - T| \geqq \int_{X'-T} h(y)dy \geqq \int_{T-X'} h(y)dy \geqq w|T - X'|;$$

$$|X' - T| + |X'T| = |X'| < t|S'| = |T| = |T - X'| + |TX'|;$$

$$|X' - T| < |T - X'| \quad \text{and} \quad w|X' - T| \geqq w|T - X'|;$$

$$w \leqq 0, \quad A(w) = S'; \qquad |S'| = |A(w)| \leqq t|S'|;$$

$$t \geqq 1.$$

But $t < 1$, so that $|X'| \geqq t|S'|$.

4.2 THEOREM. *If B is the boundary*(10) *of the open*(11) *set $A \subset E_n$, $\nu(A, x) \neq \theta$ and $D_\Phi^{\triangle}(B, x) \leqq 1$, then $D_\Phi^{\nabla}(B, x) = 1$ and $\nu(A, x)$ is Φ perpendicular to B at x.*

Proof. Since all the notions involved are invariant under distance preserving transformations of E_n, we may assume $x = \theta$ and $\nu(A, x) = i$. It will be sufficient to show that $D_\Phi^{\nabla}(B, \theta) = 1$ and

$$\operatorname{dir}_\Phi (B, \theta) \subset \underset{y}{E} [|y_n| < \epsilon] \quad \text{for every} \quad \epsilon > 0.$$

Let $\epsilon > 0$ be given and take

$$\beta = B \underset{z}{E} [|z_n| < \epsilon |z|].$$

Since $D_\Phi^{\triangle}(B, \theta) \leqq 1$, we shall complete the proof by showing that $D_\Phi^{\nabla}(\beta, \theta) \geqq 1$.

(10) We say T is the boundary of S if and only if $T = [\text{closure } S] \cdot [\text{closure } (E_n - S)]$.

(11) The hypothesis that A be open is unnecessary. This may be seen by applying the theorem to the open set $A_1 = \text{Int } A = (A + B) - B$, whose boundary is a subset of B, and for which $\nu(A_1, x) = \nu(A, x)$.

For this purpose choose a number t such that $0<t<1$. For each $r>0$ let

$$S_r^+ = K_x^r \underset{z}{E}\,[0 < z_n < \epsilon|z|], \qquad S_r^- = K_x^r \underset{z}{E}\,[0 < -z_n < \epsilon|z|],$$

$$C_r = (S_r^+)' = (S_r^-)'$$

and use lemma 4.1 to select a number u such that $0<u<1$ and $|X'| \geqq t|C_1|$ whenever X is either a subset of S_1^+ of class F_σ with $|X| \geqq u|S_1^+|$, or a subset of S_1^- of class F_σ with $|X| \geqq u|S_1^-|$.

Use 3.6 to choose $\rho>0$ so that $0<r<\rho$ implies

$$|S_r^+ - A| \geqq u|S_r^+| \quad \text{and} \quad |S_r^-A| \geqq u|S_r^-|.$$

Now fix such a number r and denote

$$X^+ = \underset{z}{E}\,[rz \in (S_r^+ - A)], \qquad X^- = \underset{z}{E}\,[rz \in S_r^-A].$$

We see that

$$|X^+| = r^{-n}|S_r^+ - A| \geqq ur^{-n}|S_r^+| = u|S_1^+|;$$

hence

$$|(S_r^+ - A)'| = r^{n-1}|(X^+)'| \geqq tr^{n-1}|C_1| = t|C_r|;$$

similarly $|(S_r^-A)'| \geqq t|C_r|$, and consequently

$$\begin{aligned} |(S_r^+ - A)'(S_r^-A)'| &= |(S_r^+ - A)'| + |(S_r^-A)'| - |(S_r^+ - A)' + (S_r^-A)'| \\ &\geqq 2t|C_r| - |C_r| = (2t-1)|C_r|. \end{aligned}$$

But $(S_r^+-A)'(S_r^-A)' \subset (\beta K_x^r)'$, because every line segment which joins a point of A to a point of $E_n - A$ must have a point in common with B. Thus

$$\Phi(\beta K_x^r) \geqq |(\beta K_x^r)'| \geqq (2t-1)|C_r| = (2t-1)\gamma(K_x^r).$$

Since r was an arbitrary positive number less than ρ, we infer that

$$D_\Phi^\triangledown(\beta, x) \geqq 2t - 1.$$

Let $t \to 1$.

4.3 THEOREM. *If X is a Lebesgue measurable subset of E_n, $Y \subset E_{n-1}$, and g is such a Lipschitzian function on E_{n-1} to E_n that*

$$g^*(Y) \subset \Lambda^+(X)$$

and

$$[g(y)]' = y \quad \textit{for} \quad y \in E_{n-1},$$

then $\tau g(y)$ points into X at $g(y)$ for almost all y in Y:

Proof. Let

$$T_k = [\overset{+}{\Lambda}_k(X)]' \cdot Y \quad \text{for} \quad k = 1, 2, 3, \cdots$$

and check that

$$Y = [g^*(Y)]'Y \subset [\Lambda^+(X)]'Y = \sum_{k=1}^{\infty} T_k.$$

Since g is differentiable almost everywhere in E_{n-1}, and T_k has density 1 at almost all of its points, we shall complete the proof by verifying the following.

Statement. If k is a positive integer, $y \in T_k$, g is differentiable at y, and T_k has density 1 at y, then $\tau g(y)$ points into X at $g(y)$.

Proof. Let L be the differential of g at y and denote $c = \tau g(y)$. Obviously $c_n Jg(y) = 1$. Let $x = g(y)$ and

$$H_r = K_x^r \underset{z}{E} [(z - x) \cdot c > 0] \quad \text{for} \quad r > 0.$$

Choose $\eta > 0$.

Select $\epsilon > 0$ so that $|H_1 - U_1| \leqq \eta |H_1|$, where

$$U_r = K_x^r \underset{z}{E} [(z - x) \cdot c > \epsilon |z - x|] \quad \text{for} \quad r > 0,$$

and then take $\rho > 0$ so that $k\rho(1 + \epsilon + \|L\|) < 1$ and

$$|g(w) - g(y) - L(w - y)| \leqq \epsilon |w - y| \quad \text{whenever} \quad |w - y| < \rho.$$

Now suppose $0 < r < \rho$.

We first prove:

(1) If $w \in T_k$ and $(w \circ t) \in U_r$, then $(w \circ t) \in X$.

In fact the hypotheses of (1) imply that $|w - y| \leqq |z - x| < r$, where $z = (w \circ t)$, and that

$$\begin{aligned} k^{-1} &> (1 + \epsilon + \|L\|)r \geqq |z - x| + |g(y) - g(w)| \geqq |z - g(w)| \geqq [z - g(w)]_n \\ &= Jg(y)[z - g(w)] \cdot c = Jg(y)[(z - x) \cdot c - [g(w) - g(y)] \cdot c] \\ &> Jg(y)[\epsilon |z - x| + |c \cdot L(w - y)| - \epsilon |w - y|] \\ &= Jg(y)[|z - x| - |w - y|]\epsilon \geqq 0. \end{aligned}$$

Since $z_n = t$, this implies $k^{-1} > t - [g(w)]_n > 0$. But $g(w) \in \Lambda_k^+(X)$, so that $(w \circ t) \in X$. Thus (1) is proved.

Next we define the set W as follows:

$w \in W$ if and only if $w \in U_r'$ and

$$|E_1 \underset{t}{E} [(w \circ t) \in U_r - X]| = 0.$$

From the Fubini Theorem we know that W is Lebesgue measurable, and from (1) we infer $U_r' T_k \subset W$. Hence

$$|U_r' - W| = |U_r'| - |W| \leqq |U_r'| - |U_r' T_k|,$$

and we apply Fubini's Theorem again to obtain

$$|U_r - X| \leqq (\text{diam } U_r)\cdot |U_r' - W| \leqq 2r(|U_r'| - |U_r' T_k|).$$

Consequently

$$\frac{|H_r - X|}{|H_r|} \leqq \frac{|H_r - U_r|}{r^n |H_1|} + \frac{|U_r - X|}{r^n |H_1|}$$

$$\leqq \frac{|H_1 - U_1|}{|H_1|} + \frac{2r|U_r'|}{r^n |H_1|}\cdot\frac{|U_r'| - |U_r' T_k|}{|U_r'|},$$

$$\frac{|H_r - X|}{|H_r|} \leqq \eta + \frac{2|U_1'|}{|H_1|}\left(1 - \frac{|U_r' T_k|}{|U_r'|}\right).$$

Remembering that r was an arbitrary positive number less than ρ, and that T_k has density 1 at y, we conclude

$$\limsup_{r\to 0+} \frac{|H_r - X|}{|H_r|} \leqq \eta.$$

Since $\eta > 0$ was freely chosen, the last relation implies

$$\lim_{r\to 0+} \frac{|H_r - X|}{|H_r|} = 0,$$

so that c points into X at x.

4.4 THEOREM. *If X is a Lebesgue measurable subset of E_n, $Y \subset E_{n-1}$, and g is such a Lipschitzian function on E_{n-1} to E_n that*

$$g^*(Y) \subset \Lambda^-(X)$$

and

$$[g(y)]' = y \quad \textit{for} \quad y \in E_{n-1},$$

then $-\tau g(y)$ points into X at $g(y)$ for almost all y in Y.

4.5 THEOREM. *If $A \subset E_n$ and f is the function on E_n to E_n such that $f(x) = \nu(A, x)$ for $x \in E_n$, then f is a Borel measurable function*(12).

Proof. Let

$$S = E_n \mathop{E}_{u} [|u| = 1].$$

(12) By this we mean that the counter-image of every closed set is a Borel set. This condition is satisfied if and only if each of the coordinate functions $f_1, f_2, \cdots, f_n$, which are defined by the relation $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$, is a Borel measurable function in the classical sense.

Since the range of f is a subset of $S+\{\theta\}$, it is sufficient to prove:

If C is a closed subset of S and

$$G = \underset{x}{E}\,[f(x) \in C],$$

then G is a Borel set.

Suppose C and G are so fixed and denote

$$R = E_n \times S = \underset{(x,u)}{E}\,[x \in E_n \text{ and } u \in S],$$

$$H_r(x, u) = K_x^r \underset{z}{E}\,[(z - x)\cdot u > 0] \quad \text{for} \quad r > 0, \qquad (x, u) \in R.$$

For each $r>0$ let g_r be the function on R such that

$$g_r(x, u) = \frac{|H_r(x, u)A| + |H_r(x, -u) - A|}{|K_x^r|} \quad \text{for} \quad (x, u) \in R.$$

We denote the symmetric difference of two sets X and Y by

$$\langle X, Y\rangle = (X - Y) + (Y - X)$$

and infer that

$$\text{(1)} \qquad |g_r(x, u) - g_r(y, v)| \leqq \frac{|\langle H_r(x, u), H_r(y, v)\rangle| + |\langle H_r(x, -u), H_r(y, -v)\rangle|}{|K_x^r|}$$

for $r>0$, $(x, u)\in R$, $(y, v)\in R$.

For each positive integer j let

$$T_j = \sum_{u\in C} \sum_{k=1}^{\infty} E_n \underset{x}{E}\,[0 < r < k^{-1} \text{ implies } g_r(x, u) \leqq j^{-1}],$$

and let

$$T = \prod_{j=1}^{\infty} T_j.$$

The remainder of the proof is divided into two parts.

Part 1. If j is a positive integer, then T_j is of class F_σ.

Proof. It follows from (1) that g_r is continuous for each $r>0$. Consequently the set

$$Q_k = \prod_{0<r<k^{-1}} R \underset{(x,u)}{E}\,[g_r(x, u) \leqq j^{-1}]$$

is closed for each positive integer k. Likewise closed is

$$H = R \underset{(x,u)}{E}\,[u \in C],$$

and, letting P be the function on R to E_n such that

$$P(x, u) = x \quad \text{for} \quad (x, u) \in R,$$

we easily see that

$$T_j = \sum_{k=1}^{\infty} P^*(HQ_k).$$

But P projects sets of class F_σ into sets of class F_σ.

Part 2. $G=T$.

Proof. Clearly

$$G = E_n \mathop{E}_{x} \left[\lim_{r\to 0+} g_r(x, u) = 0 \text{ for some } u \in C \right],$$

$$T = E_n \mathop{E}_{x} \left[\inf_{u\in C} (\limsup_{r\to 0+} g_r(x, u)) = 0 \right].$$

Now fix a point $x \in E_n$ and let

$$h_r(u) = g_r(x, u), \qquad h(u) = \limsup_{r\to 0+} h_r(u) \quad \text{for} \quad u \in S.$$

From (1) we infer that

$$| h_r(u) - h_r(v) | \leqq \frac{| \langle H_1(x, u), H_1(x, v)\rangle | + | \langle H_1(x, -u), H(x, -v)\rangle |}{| K_x^1 |}$$

whenever $u \in S$, $v \in S$. Thus the functions h_r are equicontinuous. Hence h is continuous and assumes its minimum on the compact set C. Consequently

$$T = E_n \mathop{E}_{x} \left[\limsup_{r\to 0+} g_r(x, u) = 0 \text{ for some } u \in C \right],$$

which implies $T = G$.

5. **Restrictedness.** The following two theorems are analogous to some results of §7 of RM. Since the proof of our theorems is an almost literal repetition of arguments used by Morse and Randolph, we merely illustrate what sort of changes have to be made:

Replace L, ϕ, ϕ^0 by Φ, and judiciously replace $2r$ by $\gamma(K_x^r)$. Use the relation $x = (x' \circ x_n)$ for $x \in E_n$, instead of the relation $x = (x_1, x_2)$ for $x \in E_2$, for instance

$$\langle x, a, \lambda \rangle = \mathop{E}_{y} \left[| y' - x' | < a \text{ and } | y_n - x_n | < \lambda a \right],$$

$$| x | \leqq | x' | + | x_n |, \qquad (x - x_n i)_j = (x')_j$$

for $x \in E_n$ and $j = 1, 2, \cdots, n-1$.

5.1 Theorem. *If* $S \subset E_n$, $\Phi(S) < \infty$ *and*

$$D_\Phi^{\triangle}(S, x) > 0, \qquad (\{i\} + \{-i\}) \operatorname{dir}_\Phi (S, x) = 0$$

for Φ *almost all* x *in* S, *then there are Borel sets* $T_1, T_2, T_3, \cdots \subset E_{n-1}$ *and Lipschitzian functions* $g_1, g_2, g_3, \cdots$ *on* E_{n-1} *to* E_n *such that*

$$\Phi\left[S - \sum_{k=1}^{\infty} g_k^*(T_k)\right] = 0,$$

$$[g_k(y)]' = y \quad for \quad y \in E_{n-1}, \qquad k = 1, 2, 3, \cdots.$$

5.2 THEOREM. *If $S \subset E_n$, $\Phi(S) < \infty$ and S is Φ restricted at Φ almost all of its points, then there are Borel sets $T_1, T_2, T_3, \cdots \subset E_{n-1}$ and Lipschitzian functions $g_1, g_2, g_3, \cdots$ on E_{n-1} to E_n such that*

$$\Phi\left[S - \sum_{k=1}^{\infty} g_k^*(T_k)\right] = 0.$$

5.3 THEOREM. *If $\Phi(S) < \infty$, T is a Borel set of E_{n-1} and f is such a Lipschitzian function on E_{n-1} to E_n that $f^*(T) \subset S$, then there is such a Borel set $A \subset T$ that*

$$\Phi[f^*(T) - f^*(A)] = 0,$$

and $\tau f(y)$ is Φ perpendicular to S at $f(y)$ whenever $y \in A$.

Proof. Let

$$A_1 = T \mathop{E}_{y} [D_\Phi^\triangle(S, f(y)) \leqq 1],$$
$$A_2 = A_1 \mathop{E}_{y} [f \text{ is differentiable at } y \text{ with } Jf(y) > 0],$$
$$A_3 = A_2 \mathop{E}_{y} [Jf \text{ is approximately continuous at } y],$$

and note that A_1, A_2, A_3 are measurable sets. Choose a Borel set $A_4 \subset A_3$ for which $|A_3 - A_4| = 0$, and use 3.15 of this paper and 4.5 of SA II to check

$$\Phi[f^*(T) - f^*(A_1)] = 0, \qquad \Phi[f^*(A_1 - A_2)] = 0,$$
$$|A_2 - A_4| = 0 = \Phi[f^*(A_2 - A_4)], \qquad \Phi[f^*(T) - f^*(A_4)] = 0.$$

Next select(13) a Borel set $A_5 \subset A_4$ such that $f^*(A_5) = f^*(A_4)$ and f is univalent on A_5. Finally let A be the set of those points of A_5 at which A_5 has Lebesgue density 1. We see that $A \subset T$, A is a Borel set and

$$|A_5 - A| = 0 = \Phi[f^*(A_5 - A_4)], \qquad \Phi[f^*(T) - f^*(A)] = 0.$$

Now pick a point $y \in A$. Abbreviate $x = f(y)$, $c = \tau f(y)$. In order to show that c is Φ perpendicular to S at x, we proceed as in the proof of 4.2: Let $\epsilon > 0$ be given, take

$$\beta = S \mathop{E}_{z} [|(z - x) \cdot c| \leqq \epsilon |z - x|],$$

and complete the proof by verifying (1) below.

For this purpose choose a number t so that $0 < t < 1$.

(13) See H. Federer and A. P. Morse, *Some properties of measurable functions*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 276, Theorem 5.1.

Let L be the differential of f at y, let M be the inverse of L, and choose a number η so that

$$0 < \eta \leqq \epsilon(\|M\|^{-1} - \eta) \quad \text{and} \quad (1 + \eta\|M\|)^{n-1}t \leqq 1.$$

Abbreviate $s = (1+\eta\|M\|)^{-1}$ and select $\delta > 0$ so that $|f(w) - f(y) - L(w-y)| \leqq \eta|w-y|$ whenever $|w-y| \leqq \delta$.

We define

$$U_r = E_{n-1} \underset{w}{E} [x + L(w - y) \in K_x^r] \quad \text{for} \quad r > 0,$$

and note that diam $U_r \leqq 2\|M\|r$.

Now suppose $0 < 2\|M\|r \leqq \delta$ and $w \in A\,U_{rs}$, $z = f(w)$. Then $|w-y| \leqq 2\|M\|rs \leqq \delta s < \delta$ and $|L(w-y)| \leqq rs$. Hence

$$|z - x| \leqq rs + \eta|w - y| \leqq rs + \eta|M[L(w - y)]| \leqq rs(1 + \eta\|M\|) = r$$

and $z \in K_x^r$. Furthermore

$$\begin{aligned} |z - x| &\geqq |L(w - y)| - \eta|w - y| \geqq \|M\|^{-1}|w - y| - \eta|w - y| \\ &= (\|M\|^{-1} - \eta)|w - y|, \\ |c\cdot(z - x)| &= |c\cdot L(w - y) + c\cdot[z - x - L(w - y)]| \\ &= |c\cdot[z - x - L(w - y)]| \leqq |z - x - L(w - y)| \\ &\leqq \eta|w - y| \leqq \eta(\|M\|^{-1} - \eta)^{-1}|z - x| \leqq \epsilon|z - x|, \end{aligned}$$

which implies $z \in \beta$. Accordingly $z \in \beta K_x^r$.

We have proved that

$$f^*(A\,U_{rs}) \subset \beta K_x^r \quad \text{for} \quad 0 < r < \delta(2\|M\|)^{-1}.$$

For all such r we have

$$\Phi(\beta K_x^r) \geqq \Phi[f^*(A\,U_{rs})] = \int N(f, A\,U_{rs}, z)d\Phi z = \int_{AU_{rs}} Jf(w)dw,$$

$$\gamma(K_x^r) = |U_r|\,Jf(y) = s^{1-n}|U_{rs}|\,Jf(y),$$

$$\frac{\Phi(\beta K_x^r)}{\gamma(K_x^r)} \geqq s^{n-1}\frac{|A\,U_{rs}|}{|U_{rs}|}\,\frac{\int_{AU_{rs}} Jf(w)dw}{Jf(y)\,|A\,U_{rs}|}.$$

Since A has density 1 at y and Jf is approximately continuous at y we conclude

$$D_\Phi^\triangledown(\beta, x) \geqq s^{n-1} = (1 + \eta\|M\|)^{1-n} \geqq t.$$

From the arbitrary nature of t we finally infer

(1) $$D_\Phi^\triangledown(\beta, x) \geqq 1.$$

The proof is complete.

5.4 *Remark.* If y is any point of the set A of Theorem 5.3, then

$$\text{dir}_\Phi (S, f(y)) = E_n \underset{u}{E} [|u| = 1 \text{ and } u \cdot \tau f(y) = 0].$$

This fact will not be used in this paper.

5.5 THEOREM. *If $Z \subset E_n$, $\Phi(Z) < \infty$ and Z is Φ restricted at Φ almost all of its points with*

$$(\{i\} + \{-i\}) \,\text{dir}_\Phi (Z, x) \neq 0 \quad \textit{for} \quad x \in Z,$$

then $|Z'| = 0$.

Proof. In view of 5.2 we assume that we are dealing with the special case in which $Z = f^*(T)$, where T is a Borel set of E_{n-1} and f is a Lipschitzian function on E_{n-1} to E_n. We select A in accordance with 5.3, here $S = Z$, and define the function g on E_{n-1} to E_{n-1} by the relation

$$g(y) = [f(y)]' \quad \text{for} \quad y \in E_{n-1}.$$

Evidently g is Lipschitzian and

$$|Z'| \leqq |g^*(A)| + \Phi[Z - f^*(A)] \leqq \int N(g, A, w)dw = \int_A Jg(y)dy.$$

We shall complete the proof by showing that $Jg(y) = 0$ for every $y \in A$.

Pick $y \in A$ and let L be the approximate differential of f at y. Since $\tau f(y) \in E_n$ we know that $Jf(y) > 0$ and the points $L^1, L^2, \cdots, L^{n-1}$ are linearly independent. Using 5.3, our present hypotheses, and 3.15, we see that

$$\tau f(y) \cdot i = 0 \quad \text{and} \quad \tau f(y) \cdot L^j = 0$$

for $j = 1, 2, \cdots, n-1$. Hence the points $i, L^1, L^2, \cdots, L^{n-1}$ are linearly dependent. Since the last $n-1$ are independent, we can find numbers $t_1, t_2, \cdots, t_{n-1}$ not all zero such that

$$\sum_{j=1}^{n-1} t_j L^j = i.$$

But projection is a linear operation and

$$\sum_{j=1}^{n-1} t_j (L^j)' = i' = (0, \cdots, 0) \in E_{n-1}.$$

This means that the columns $(L^1)', \cdots, (L^{n-1})'$ of the approximate differential of g at y are linearly dependent, hence $Jg(y) = 0$.

6. The Gauss-Green Theorem.

6.1 DEFINITION. For $j = 1, 2, \cdots, n$ and $S \subset E_n$ we define the set $\Omega_j(S)$ as follows:

$f \in \Omega_j(S)$ if and only if f is a numerically valued function, S is a subset of the domain of f,

$$-\infty < \int f(x)\nu_j(S, x)d\Phi x < \infty\,(^{14}), \quad -\infty < \int_S D_j f(x)dx < \infty,$$

and there is a set $V \subset E_{n-1}$ such that $|V| = 0$ and

$$f(y_1, \cdots, y_{j-1}, b, y_j, \cdots, y_{n-1}) - f(y_1, \cdots, y_{j-1}, a, y_j, \cdots, y_{n-1})$$
$$= \int_a^b D_j f(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1})dt$$

whenever $y \in E_{n-1} - V$, $a < b$, and $(y_1, \cdots, y_{j-1}, t, y_j, \cdots, y_{n-1}) \in S$ for $a \leqq t \leqq b$.

6.2 *Notation.*

$$X^y = E_1 \mathop{E}_t [(y \circ t) \in X] \quad \text{for} \quad X \subset E_n,\ y \in E_{n-1}.$$

Observe that in case $j = n$ the last condition in 6.1 can be stated as follows:

$$f(y \circ b) - f(y \circ a) = \int_a^b D_n f(y \circ t)dt$$

whenever $y \in E_{n-1} - V$, $a < b$, and $[a, b] \subset S^y$.

6.3 *Remark.* In connection with Definition 6.1 we remind the reader that the condition

$$(1) \qquad -\infty < \int f(x)\nu_j(S, x)d\Phi x < \infty$$

does *not* imply that the exterior normal of S exists anywhere. In fact 3.6 tells us that $\nu_j(S, x) = 0$ whenever S has no exterior normal at x.

If T is the boundary[15] of S, then the relation

$$-\infty < \int_T f(x)d\Phi x < \infty$$

is *sufficient* for condition (1). This follows from Theorem 4.5 and the fact that

$$|\nu(S, x)| \leqq 1 \quad \text{for} \quad x \in E_n, \qquad \nu(S, x) = \theta \quad \text{for} \quad x \in E_n - T.$$

6.4 THEOREM. *If*

(I) *A is a bounded open subset of E_n, B is the boundary*[15] *of A, and $\Phi(B) < \infty$;*

(II) *R is the set of all points at which B is Φ restricted,*

$$C^+ = \Lambda^+(E_n - A)\Lambda^-(A), \qquad C^- = \Lambda^+(A)\Lambda^-(E_n - A),$$
$$C = C^+ + C^-, \qquad F = E_n \mathop{E}_x [B^{x'} \textit{ is finite}],$$

and

(14) $\nu_j(S, x)$ is the jth coordinate of $\nu(S, x)$.

(15) See footnote 10.

$$|(CF-R)'|\neq 0;$$

(III) $f\in\Omega_n(A+B)$;

then

$$\int_A D_nf(x)dx=\int f(x)\nu_n(A,x)d\Phi x.$$

Proof. Select $V\subset E_{n-1}$ in accordance with (III) and 6.1. Let

$$H=B\underset{x}{E}[\nu_n(A,x)\neq 0],$$
$$W=RB\underset{x}{E}[(\{i\}+\{-i\})\operatorname{dir}_\Phi(B,x)=0],$$
$$Z=RB\underset{x}{E}[(\{i\}+\{-i\})\operatorname{dir}_\Phi(B,x)\neq 0].$$

Use 5.1 to select Borel sets $T_1, T_2, T_3, \cdots \subset E_{n-1}$, Lipschitzian functions $g_1, g_2, g_3, \cdots$ on E_{n-1} to E_n, and *disjoint* sets $S_1, S_2, S_3, \cdots$ such that $S_k=g_k^*(T_k)$, $[g_k(y)]'=y$ for $k=1, 2, 3, \cdots, y\in E_{n-1}$, and

$$\Phi(W-S)=0 \quad\text{with}\quad S=\sum_{k=1}^{\infty}S_k.$$

Next we define the set G:

$$x\in G \text{ if and only if } x\in F,\ x'\in E_{n-1}-V,\ C^{x'}\subset S^{x'}$$

and $k=1\ 2\ 3, \cdots$ implies

$$\nu(A,x'\circ t)=\tau g_k(x') \qquad\text{whenever}\quad t\in(S_kC^+)^{x'},$$
$$\nu(A,x'\circ t)=-\tau g_k(x') \qquad\text{whenever}\quad t\in(S_kC^-)^{x'},$$
$$\nu(A,x'\circ t)=\theta \qquad\text{whenever}\quad t\in(S_k-C)^{x'}.$$

Let h_k be the characteristic function of HGS_k, and let

$$p_k(x)=h_k(x)f(x)\nu_n(A,x)\quad\text{for}\quad k=1,2,3,\cdots, x\in E_n,$$
$$\alpha(y)=\sum_{k=1}^{\infty}p_k[g_k(y)]Jg_k(y)\quad\text{for}\quad y\in E_{n-1},$$
$$\beta(y)=\int_{A^y}D_nf(y\circ t)dt\quad\text{for}\quad y\in E_{n-1},$$

The remainder of the proof is divided into eleven parts.

Part 1. If $X\subset W$ with $|X'|=0$, then $\Phi(X)=0$.

Proof. $X=(X-S)+\sum_{k=1}^{\infty}XS_k\subset(W-S)+\sum_{k=1}^{\infty}g_k^*(X')$, $\Phi(W-S)=0$, and g_k is Lipschitzian. Use Lemma 4.1 of SA II.

Part 2. $|E_{n-1}-F'|=0$.

Proof. Letting P be the projecting function such that

$$P(x) = x' \quad \text{for} \quad x \in E_n,$$

we deduce from Theorem 4.4 of SA I and Remark 2.3 of SA II that

$$\int N(P, B, y)dy \leqq \Phi(B) < \infty.$$

Hence $N(P, B, y) < \infty$ for almost all y in E_{n-1}, which implies that $y \in F'$ for almost all y in E_{n-1}.

Part 3. $|(C-WS)'| = 0$.

Proof. $(C-WS)' \subset (C-W)' + (W-S)' \subset (C'-F') + (C-W)'F' + (W-S)' = (C'-F') + (CF-W)' + (W-S)' \subset (E_{n-1}-F') + (CF-R)' + Z' + (W-S)'$.

Apply Part 2, (II), Theorem 5.5, and the relation $\Phi(W-S) = 0$.

Part 4. $\Phi(W-G) = 0$.

Proof. For each positive integer k let

$$\begin{aligned} U_k = \underset{y}{E}\, &[g_k(y) \in C^+ \text{ and } \nu(A, g_k(y)) \neq \tau g_k(y)] \\ &+ \underset{y}{E}\, [g_k(y) \in C^- \text{ and } \nu(A, g_k(y)) \neq -\tau g_k(y)] \\ &+ \underset{y}{E}\, [g_k(y) \in (F - C) \text{ and } \nu(A, g_k(y)) \neq \theta]. \end{aligned}$$

In view of the definition of C^+ and C^-, and of the relation

$$(F - C) \subset \Lambda^+(A)\Lambda^-(A) + \Lambda^+(E_n - A)\Lambda^-(E_n - A),$$

we can apply Theorems 4.3 and 4.4 to each of the three sets whose union is U_k, and infer

$$|U_k| = 0 \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

But it follows from the definition of G that

$$(W - G)' \subset (E_{n-1} - F') + V + (C - S)' + \sum_{k=1}^{\infty} U_k.$$

Hence Parts 2 and 3 imply

$$|(W - G)'| = 0.$$

Use Part 1 to complete the proof.

Part 5. $|C'-G'| = 0$.

Proof. Apply Parts 3 and 4 to the relation $(C'-G') \subset (C-G)' \subset (C-W)' + (W-G)'$.

Part 6. $|A'-G'| = 0$.

Proof. $(A'-G') \subset (E_{n-1}-F') + (A'F'-C') + (C'-G')$.

From parts 2 and 5 we infer that the first and third of the last three sets each has measure zero. We shall complete the proof by showing that the second set is vacuous.

For this purpose we suppose $y \in (A'F'-C')$. Then A^y is the union of a *finite* number of *nonvacuous* open intervals, and $(y \circ \sup A^y) \in C^+$. Hence $y \in (C' - C') = 0$.

Part 7. $\alpha(y) = \beta(y)$ for almost all y in G'.

Proof. Note that

$$\left| E_{n-1} - \prod_{k=1}^{\infty} (\text{domain } Jg_k) \right| = 0,$$

and let y be a point for which

$$y \in G' \prod_{k=1}^{\infty} (\text{domain } Jg_k).$$

Hence $y \in F'$, B^y is finite, and there is a finite, disjointed family Q of nonvacuous open intervals such that

$$A^y = \sigma(Q).$$

Letting

$$\lambda = \sum_{I \in Q} \{\inf I\}, \qquad \rho = \sum_{I \in Q} \{\sup I\},$$

we see from the definition of C^+ and C^- that

$$\rho - \lambda = (C^+)^y, \qquad \lambda - \rho = (C^-)^y.$$

Remembering that $y \in E_{n-1} - V$, we compute:

$$\beta(y) = \sum_{t \in \rho - \lambda} f(y \circ t) - \sum_{t \in \lambda - \rho} f(y \circ t) = \sum_{t \in (C^+)^y} f(y \circ t) - \sum_{t \in (C^-)^y} f(y \circ t).$$

Since $y \in G'$, we know that

$$(C^+)^y = S^y(C^+)^y = (SC^+)^y = \sum_{k=1}^{\infty} (S_k C^+)^y,$$

$$(C^-)^y = \sum_{k=1}^{\infty} (S_k C^-)^y.$$

Hence

$$\beta(y) = \sum_{k=1}^{\infty} \sum_{t \in (S_k C^+)^y} f(y \circ t) - \sum_{k=1}^{\infty} \sum_{t \in (S_k C^-)^y} f(y \circ t).$$

We next use the definition of G, and 3.15, to obtain the propositions:

$t \in (S_k C^+)^y$ implies $\nu_n(A, y \circ t) Jg_k(y) = [\tau g_k(y)]_n Jg_k(y) = 1$,

$t \in (S_k C^-)^y$ implies $\nu_n(A, y \circ t) Jg_k(y) = -[\tau g_k(y)]_n Jg_k(y) = -1$,

$t \in (S_k - C)^y$ implies $\nu_n(A, y \circ t) Jg_k(y) = 0$.

Consequently

$$\beta(y) = \sum_{k=1}^{\infty} \sum_{t \in (S_k)^y} f(y \circ t) \nu_n(A, y \circ t) Jg_k(y).$$

Next we notice that

$$\nu_n(A, y \circ t) = 0 \quad \text{for} \quad t \in E_1 - H^y,$$

and that $y \in G'$ implies $G^y = E_1$. From these relations, and the identity $H^y G^y (S_k)^y = (HGS_k)^y$, we infer

$$\beta(y) = \sum_{k=1}^{\infty} \sum_{t \in (HGS_k)^y} f(y \circ t)\nu_n(A, y \circ t) J g_k(y).$$

Recalling the definition of h_k, and the relation

$$(y \circ t) = g_k(y) \quad \text{for} \quad t \in (S_k)^y,$$

we perform the summation with respect to t, and obtain

$$\beta(y) = \sum_{k=1}^{\infty} h_k[g_k(y)] f[g_k(y)] \nu_n[A, g_k(y)] J g_k(y)$$

$$= \sum_{k=1}^{\infty} p_k[g_k(y)] J g_k(y) = \alpha(y).$$

Part 8. $\alpha(y) = \beta(y)$ for almost all y in E_{n-1}.

Proof. Let

$$Y = E_{n-1} \mathop{E}_{y} [\alpha(y) \neq \beta(y)],$$

and check that

$$Y \subset [Y - (A' + G')] + YG' + (A' - G').$$

From Parts 7 and 6 we know that

$$|YG'| = |A' - G'| = 0.$$

Clearly $y \in (E_{n-1} - A')$ implies $A^y = 0$ and $\beta(y) = 0$.

From the relation $y \in (E_{n-1} - G')$ we can infer $G^y = 0$, $h_k[g_k(y)] = 0$ for $k = 1, 2, 3, \cdots$, and $\alpha(y) = 0$.

Hence $\alpha(y) = 0 = \beta(y)$ for $y \in [E_{n-1} - (A' + G')]$, and $[Y - (A' + G')] = 0$.

Thus $|Y| = 0$.

Part 9. $\Phi(H - W) = 0$.

Proof. From 3.16 we know that $D_{\Phi}^{\triangle}(B, x) \leq 1$ for Φ almost all x in B. Hence 4.2 enables us to conclude for Φ almost all x in H that

$$x \in R = W + Z \quad \text{and} \quad \text{dir}_{\Phi}\,(B, x) \subset \mathop{E}_{z} [z \cdot \nu(A, x) = 0].$$

But, for such a point x, the relation $x \in Z$ implies

$$0 \neq \nu_n(A, x) = i \cdot \nu(A, x) = 0.$$

Thus $x \in W$ for Φ almost all x in H.

Part 10. $\Phi(H-GS)=0$.

Proof. $(H-GS)\subset(H-W)+(W-GS)=(H-W)+(W-G)+(W-S)$. Use Parts 9 and 4, and the definition of S.

Part 11. $\int_A D_n f(x)dx=\int f(x)\nu_n(A, x)d\Phi x$.

Proof. From Part 2 and the Fubini Theorem we infer $|B|=0$. Hence $\nu(A+B, x)=\nu(A, x)$ for all x, and (III) implies

$$-\infty<\int f(x)\nu_n(A, x)d\Phi x<\infty.$$

Now use Part 10, Theorem 4.5 of SA II, and Theorem 2.1 of the present paper to compute:

$$\begin{aligned}\int f(x)\nu_n(A, x)d\Phi x &= \int_H f(x)\nu_n(A, x)d\Phi x\\ &= \int_{HGS} f(x)\nu_n(A, x)d\Phi x\\ &= \sum_{k=1}^{\infty}\int h_k(x)f(x)\nu_n(A, x)d\Phi x\\ &= \sum_{k=1}^{\infty}\int p_k(x)d\Phi x\\ &= \sum_{k=1}^{\infty}\int p_k(x)N(g_k, E_{n-1}, x)d\Phi x\\ &= \sum_{k=1}^{\infty}\int p_k[g_k(y)]Jg_k(y)dy.\end{aligned}$$

Similarly

$$\infty>\int|f(x)\nu_n(A, x)|d\Phi x=\sum_{k=1}^{\infty}\int|p_k[g_k(y)]|Jg_k(y)dy.$$

Hence we may change the order of summation and integration, and use Part 8, as follows:

$$\int f(x)\nu_n(A, x)d\Phi x=\int\sum_{k=1}^{\infty}p_k[g_k(y)]Jg_k(y)dy=\int\alpha(y)dy=\int\beta(y)dy.$$

But (III) implies

$$-\infty<\int_A D_n f(x)dx<\infty,$$

and we infer from the Fubini Theorem that

$$\int_A D_n f(x)dx = \int \beta(y)dy.$$

The proof is complete.

6.5 COROLLARY. *If*

(I) *A is a bounded open subset of* E_n, *B is the boundary of A, and* $\Phi(B) < \infty$;

(II) $$Q = E_n \mathop{E}_{x} [\nu(A, x) \neq \theta], \quad F = E_n \mathop{E}_{x} [B^{x'} \text{ is finite}],$$
$$C = \Lambda^+(E_n - A)\Lambda^-(A) + \Lambda^+(A)\Lambda^-(E_n - A),$$

and

$$| (CF - Q)' | = 0;$$

(III) $f \in \Omega_n(A + B)$;

then

$$\int_A D_n f(x)dx = \int f(x)\nu_n(A, x)d\Phi x.$$

Proof. Let R be the set of all points at which B is Φ restricted, and check that

$$(CF - R)' \subset (CF - Q)' + (Q - R)'.$$

From (I), 3.16 and 4.2 we know that $\Phi(Q - R) = 0$, and use this relation and (II) to infer

$$| (CF - R)' | = 0.$$

Reference to 6.4 completes the proof.

6.6 *Remark.* With the obvious changes in the hypotheses (II) and (III) of Theorem 6.4 (or 6.5), we obtain the formula

$$\int_A D_j f(x)dx = \int f(x)\nu_j(A, x)d\Phi x,$$

where j is an integer between 1 and n.

6.7 *Remark.* In the conditions (I) and (II) of Theorem 6.4, the sets B, R, C, F are all defined in terms of the set A. Hence (I) and (II) are essentially properties of A.

It should therefore be understood what we mean by saying that a given set A *has* (*or does not have*) *the properties* (I) *and* (II) *of* 6.4.

6.8 THEOREM. *If each of the disjoint sets* A_1 *and* A_2 *has the properties* (I) *and* (II) *of* 6.4, *and A is such an open set that*

$$(A_1 + A_2) \subset A \subset \text{closure } (A_1 + A_2),$$

then A also has the properties (I) *and* (II) *of* 6.4, *and*

$$\int f(x)\nu_n(A_1, x)d\Phi x + \int f(x)\nu_n(A_2, x)d\Phi x = \int f(x)\nu_n(A, x)d\Phi x$$

whenever f is such a numerically valued function that these integrals are finite.

Proof. Recall 6.7 and 6.4, and attach the generically obvious meaning to $B, B_1, B_2; R, R_1, R_2; C, C_1, C_2; F, F_1, F_2$. Since A_1 and A_2 have the properties (I) and (II) of 6.4, we have

$$\Phi(B_1) < \infty, \qquad \Phi(B_2) < \infty, \tag{1}$$

$$|(C_1F_1 - R_1)'| = 0, \qquad |(C_2F_2 - R_2)'| = 0. \tag{2}$$

A simple check reveals that

$$B \subset B_1 + B_2, \qquad CF_1F_2 \subset C_1 + C_2, \tag{3}$$

$$\Phi(B) \leqq \Phi(B_1) + \Phi(B_2) < \infty. \tag{4}$$

From (1) and Theorem 4.4 of SA I we infer

$$|(E_n - F_1)'| = 0, \qquad |(E_n - F_2)'| = 0, \tag{5}$$

whereas repeated application of (1), (4) and 3.16 yields $\text{dir}_\Phi(B, x) = \text{dir}_\Phi(BB_1, x) = \text{dir}_\Phi(B_1, x)$ for Φ almost all x in BB_1, $\text{dir}_\Phi(B, x) = \text{dir}_\Phi(BB_2, x) = \text{dir}_\Phi(B_2, x)$ for Φ almost all x in BB_2,

$$\Phi(BB_1R_1 - R) = 0, \qquad \Phi(BB_2R_2 - R) = 0. \tag{6}$$

Next we use (3) to check

$$\begin{aligned}(CF - R) &\subset B(C - R) \subset (BCF_1F_2 - R) + \sum_{j=1}^{2} (C - F_j)\\ &\subset \sum_{j=1}^{2} (BC_j - R) + (C - F_j) = \sum_{j=1}^{2} (BB_jC_j - R) + (C - F_j)\\ &\subset \sum_{j=1}^{2} (BB_jR_j - R) + (C_j - R_j) + (C - F_j)\\ &\subset \sum_{j=1}^{2} (BB_jR_j - R) + (C_jF_j - R) + (E_n - F_j),\end{aligned}$$

$$|(CF - R)'| \leqq \sum_{j=1}^{2} \Phi(BB_jR_j - R) + |(C_jF_j - R_j)'| + |(E_n - F_j)'|,$$

and we conclude from (6), (2), (5) that

$$|(CF - R)'| = 0.$$

From this relation and (4) it follows that A has the properties (I) and (II) of 6.4.

We see from (5) that $|B_1+B_2|=0$, hence $|A-(A_1+A_2)|=0$, and we have

$$\int_{A_1} D_n f(x)dx + \int_{A_2} D_n f(x)dx = \int_A D_n f(x)dx$$

whenever $f\in\Omega_n(A_1+B_1)\Omega_n(A_2+B_2)\Omega_n(A+B)$. For each function f of this class we therefore know from Theorem 6.4 that

$$(7) \qquad \int f(x)\nu_n(A_1, x)d\Phi x + \int f(x)\nu_n(A_2, x)d\Phi x = \int f(x)\nu_n(A, x)d\Phi x.$$

Hence (7) holds in particular whenever f has continuous partial derivatives on E_n, and we can apply standard methods of approximation to deduce that (7) is valid for every numerically valued function f for which the integrals occurring in (7) are finite.

6.9 *Remark.* Suppose condition (I) of 6.4 holds. Let $C=\Lambda^+(E_n-A)\Lambda^-(A)+\Lambda^+(A)\Lambda^-(E_n-A)$, and suppose there are Lipschitzian functions $g_1, g_2, g_3, \cdots$ on E_{n-1} to E_n such that

$$\Phi\left[C - \sum_{k=1}^{\infty}(\text{range } g_k)\right] = 0.$$

Then condition (II) of 6.4 holds, and the Gauss-Green formula is true for every function f which satisfies condition (III) of 6.4.

Hence Theorem 6.4 includes the classical Gauss-Green Theorem.

7. The Gauss-Green Theorem in the plane. Throughout this section we assume $n=2$. Hence Φ is now Carathéodory linear measure.

7.1 DEFINITION. We say x is *accessible* from S if and only if $x\in E_2$, $S\subset E_2$, and x is a limit point of a connected subset of S.

7.2 THEOREM. *If A is an open subset of E_2, B is the boundary of A, and x is accessible from both A and $[E_2-(A+B)]$, then*

$$D_\Phi^{\nabla}(B, x) \geqq 1.$$

Proof. Select connected sets α and β such that

$$\alpha\subset A, \qquad \beta\subset E_2-(A+B),$$

and x is a limit point of both α and β. Since $x\in B$, we see that $x\notin\alpha$, diam $\alpha>0$; $x\notin\beta$, diam $\beta>0$.

Let ρ be a number for which

$$0<\rho, \qquad 2\rho<\text{diam }\alpha, \qquad 2\rho<\text{diam }\beta.$$

For $r>0$ we define

$$C_r = E_2 \underset{z}{E}\,[\,|z-x|=r].$$

Obviously $0<r<\rho$ implies $\alpha C_r \neq 0$ and $\beta C_r \neq 0$; consequently BC_r has at least two elements, because every circular arc, which joins a point of A to a point of $[E_2-(A+B)]$, crosses B on the way. Letting f be the function on E_2 to E_1 such that

$$f(z) = |z - x| \quad \text{for} \quad z \in E_2,$$

we infer that

$$N(f, B, r) \geqq 2 \quad \text{for} \quad 0 < r < \rho.$$

From the well known relation([16])

$$\Phi(S) \geqq |f^*(S)| \quad \text{for} \quad S \subset E_2,$$

it follows, as in the proof of 4.3 of SA I, that

$$\Phi(BK_x^\rho) \geqq \int N(f, BK_x^\rho, r)dr.$$

Hence

$$\Phi(BK_x^\rho) \geqq \int N(f, BK_x^\rho, r)dr = \int_0^\rho N(f, B, r)dr \geqq 2\int_0^\rho dr = 2\rho = \gamma(K_x^\rho).$$

Since ρ was an arbitrary sufficiently small positive number, the proof is complete.

7.3 THEOREM. *If A is a bounded open subset of E_2, B is the boundary of A, $\Phi(B)<\infty$, j is either* 1 *or* 2, *and $f \in \Omega_j(A+B)$, then*

$$\int_A D_j f(x)dx = \int f(x)\nu_j(A, x)d\Phi x.$$

Proof. Assuming $j=2$, we use the notation of the statement of Theorem 6.4. Evidently our present hypotheses include the conditions (I) and (III). We shall complete the proof by showing that (I) implies (II) in the present case $n=2$.

Each point of CF is accessible (by straight line segments) from both A and $[E_2-(A+B)]$. Hence 7.2 implies

$$D_\Phi^{\triangledown}(B, x) \geqq 1 \quad \text{for} \quad x \in CF.$$

On the other hand we infer from 3.16 that

$$D_\Phi^{\vartriangle}(B, x) \leqq 1 \quad \text{for } \Phi \text{ almost all } x \text{ in } B.$$

Consequently

$$D_\Phi^{\triangledown}(B, x) = D_\Phi^{\vartriangle}(B, x) = 1 \quad \text{for } \Phi \text{ almost all } x \text{ in } CF.$$

Applying 3.16 once more we obtain

([16]) See the proof of Theorem 4.2 of RM.

$$D_{\Phi}^{\nabla}(CF, x) = D_{\Phi}^{\triangle}(CF, x) = 1 \quad \text{for } \Phi \text{ almost all } x \text{ in } CF.$$

We can now use Theorem 11.1 of RM to infer that CF is Φ restricted at Φ almost all of its points. Hence 3.16 implies

$$\Phi(CF - R) = 0.$$

This completes the proof of (II), and of the theorem.

7.4 *Remark.* We don't know the answer to the following question:

Is the analogue, for $n > 2$, of Theorem 7.3 true or false?

The existence of *Lebesgue spines* shows that the strict analogue of 7.2 for 3-space is false, and we are not familiar with a proof of the 3-dimensional analogue of Theorem 9.5 of RM.

7.5 *Remark.* Similar to 7.2 is the following *true* statement:

If $A \subset E_2$, B is the boundary of A, and both A and $(E_2 - A)$ have positive 2-dimensional lower Lebesgue density at x, then $D_{\Phi}^{\nabla}(B, x) > 0$.

We neither prove nor use this fact in this paper.

7.6 *Remark.* Using the terminology introduced in 6.7, we can describe the idea of the proof of 7.3 as follows:

If A is a bounded open subset of E_2, B is the boundary of A, and $\Phi(B) < \infty$, then A has the properties (I) *and* (II) *of* 6.4.

7.7 *Remark.* Let p be such a *continuous* function on E_1 to E_2 that $-\infty < s < t < \infty$ implies $p(s) = p(t)$ *if and only if* $(t - s)$ *is an integer.*

Let $B = \text{range } p$. Then B is a simple closed Jordan curve, parametrized by p. Let A be the set of those points of the plane which are "*inside*" B. Thus $x \in A$ if and only if $x \in E_2$, and there is such a positive number r that $B \subset K_x^r$ and the relation $|y| \geqq r$ implies there is *no* continuum C for which $\{x\} + \{y\} \subset C \subset E_2 - B$.

From Jordan's Theorem we know that B is the boundary of A. The following facts seem of interest in case the function p is differentiable on a subset of its domain:

(I) *If* $-\infty < s < \infty$ *and* $|p'(s)| > 0$, *then either* $\nu[A, p(s)] = -\tau p(s)$ *or* $\nu[A, p(s)] = \tau p(s)$.

(II) *If* $-\infty < s < t < \infty$, $|p'(s)| > 0$, $|p'(t)| > 0$, *then there is such a number* λ *that* $\lambda^2 = 1$ *and*

$$\nu[A, p(s)] = \lambda \tau p(s) \quad \textit{and} \quad \nu[A, p(t)] = \lambda \tau p(t).$$

In this connection we note that $|p'(s)| > 0$ implies

$$\tau p(s) = \frac{(-p_2'(s), p_1'(s))}{|p'(s)|} = \frac{ip'(s)}{|p'(s)|}.$$

Hence we see from (I) that a simple relation exists between the exterior normal of A and the derivative of p, whenever the latter has a positive modulus. We are then told by (II) that the ambiguity in sign, which is inherent in (I), is not very serious, because *the same choice of sign is correct for all points* under consideration.

The proof of (I), using Jordan's Theorem, is fairly simple. We are, however, *not* familiar with a *short* proof of (II), except for the case in which p' is continuous and $|p'(s)|>0$ for $-\infty<s<\infty$.

In this paper the statements (I) and (II) are neither proved nor used. One of their immediate implications is, however, stated for completeness:

If p satisfies a Lipschitz condition, then

$$\int f(x)\nu_j(A, x)d\Phi x = \mp \int_0^1 f[p(t)]\tau_j p(t) \mid p'(t) \mid dt$$

for $j=1, 2$, and every Φ measurable numerically valued function f.

8. **Cauchy's Theorem.** As in §7 we let $n=2$. Φ is again Carathéodory linear measure. We do not distinguish between E_2 and the finite complex plane. Hence $\nu(A, x)$ is a complex number whenever $A\subset E_2$, $x\in E_2$, and the point i, which was fixed in 3.5, has now its classical significance. To avoid ambiguity, we state:

8.1 Definition. If f is a function whose domain and range are sets of complex numbers (subsets of E_2), then f' is the function such that

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \quad \textit{for every } x.$$

We say f is *conformal at x* if and only if $f'(x)\in E_2$.

8.2 *Remark. If f is conformal at each point of the open set $A\subset E_2$, and $|f(x)|\leq M$ for $x\in A$, then $x\in A$ implies $|f'(x)|\leq 8M/\pi a$, where*

$$a = \inf_{z\in E_2-A} |z - x|.$$

As is well known, this follows immediately from Cauchy's integral formulae for a *square* with center x and side a.

8.3 Theorem. *If A is a bounded open set of complex numbers, B is the boundary of A, $\Phi(B)<\infty$, f is conformal at each point of A, and f is continuous at each point of B with respect to $(A+B)$, then*

$$i\int f(x)\nu(A, x)d\Phi x = 0.$$

Proof. Let $\epsilon>0$. Determine $\eta>0$ so that

$$5\pi[\Phi(B) + 1]\eta = \epsilon,$$

and then select a number $\delta>0$ such that $|f(z)-f(x)|\leq\eta$ whenever $x\in(A+B)$, $z\in(A+B)$, $|x-z|\leq\delta$.

From the definition of Φ we obtain open connected sets $G_1, G_2, G_3, \cdots$, each of diameter less than δ, for which

$$B \subset \sum_{j=1}^{\infty} G_j \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma(G_j) < \Phi(B) + 1.$$

Since B is compact, we use the Heine-Borel theorem to obtain an integer p such that

$$B \subset \sum_{j=1}^{p} G_j.$$

We may certainly assume $BG_j \neq 0$ for $j=1, 2, \cdots, p$.

Select points $x^j \in BG_j$ and define

$$C_j = E_2 \underset{z}{E} \left[\, | z - x^j | < \operatorname{diam} G_j \right], \quad \Gamma_j = E_2 \underset{z}{E} \left[\, | z - x^j | = \operatorname{diam} G_j \right]$$

for $j=1, 2, \cdots, p$. Clearly $G_j \subset C_j$ and

$$B \subset \sum_{j=1}^{p} C_j.$$

Now let

$$A_0 = A - \sum_{j=1}^{p} (C_j + \Gamma_j),$$

$$A_1 = AC_1 \quad \text{and} \quad A_k = AC_k - \sum_{j=1}^{k-1} (C_j + \Gamma_j) \quad \text{for} \quad k = 2, 3, \cdots, p,$$

and denote

$$B_j = \text{boundary } A_j, \qquad H_j = E_2 \underset{x}{E} \left[\nu(A_j, x) \neq \theta \right]$$

for $j=0, 1, 2, \cdots, p$.

The theorem follows from the last of the six parts into which we divide the remainder of the proof.

Part 1. $B_k \subset B + \sum_{j=1}^{p} \Gamma_j$ and $\Phi(B_k) < \infty$ for $k=0, 1, 2, \cdots, p$.

Proof. If $X + Y \subset E_2$, then

$$\operatorname{Bdr}(X + Y) \subset \operatorname{Bdr} X + \operatorname{Bdr} Y, \qquad \operatorname{Bdr}(E_2 - X) = \operatorname{Bdr} X.$$

Setting $C_0 = A$, we hence see that $k=0, 1, 2, \cdots, p$ implies

$$B_k = \operatorname{Bdr} A_k \subset \operatorname{Bdr} A + \operatorname{Bdr} C_k + \sum_{j=1}^{p} \operatorname{Bdr}(C_k + \Gamma_k) \subset B + \sum_{j=1}^{p} \Gamma_j,$$

$$\Phi(B_k) \leq \Phi(B) + \sum_{j=1}^{p} \pi \operatorname{diam} \Gamma_j.$$

Part 2. If $0 \leq j < k < l \leq p$ are integers, then $H_j H_k H_l = 0$.

Proof. Otherwise we could pick a point $x \in H_j H_k H_l \neq 0$ and use the fact that the set A_j, A_k, A_l are *disjoint*, together with the definition of the exterior normal, to infer

$$1 = \lim_{r\to 0+} \frac{|K_x^r|}{|K_x^r|}$$

$$\geqq \liminf_{r\to 0+} \frac{|K_x^r A_j|}{|K_x^r|} + \liminf_{r\to 0+} \frac{|K_x^r A_k|}{|K_x^r|} + \liminf_{r\to 0+} \frac{|K_x^r A_l|}{|K_x^r|} \geqq \frac{3}{2}.$$

Part 3. $\eta \sum_{j=0}^{p} \Phi(H_j) \leqq \epsilon$.

Proof. Let h_k be the characteristic function of H_k. Then Part 2 implies

$$\sum_{j=0}^{p} h_j(x) \leqq 2 \quad \text{for} \quad x \in E_2.$$

Use this relation and Part 1 to check:

$$\sum_{j=0}^{p} \Phi(H_j) = \sum_{j=0}^{p} \int h_j(x) d\Phi x \leqq 2\Phi\Big(\sum_{j=0}^{p} H_j\Big) \leqq 2\Phi\Big(\sum_{j=0}^{p} B_j\Big)$$

$$\leqq 2\Phi\Big(B + \sum_{j=1}^{p} \Gamma_j\Big) \leqq 2\Phi(B) + 2\pi \sum_{j=1}^{p} \operatorname{diam} \Gamma_j$$

$$= 2\Phi(B) + 4\pi \sum_{j=1}^{p} \operatorname{diam} G_j = 2\Phi(B) + 4\pi \sum_{j=1}^{p} \gamma(G_j)$$

$$< 2\Phi(B) + 4\pi[\Phi(B) + 1] \leqq 5\pi[\Phi(B) + 1].$$

Multiply by η to complete the proof of Part 3.

Part 4. $\left|\int f(x)\nu(A_j, x)d\Phi x\right| \leqq \eta \Phi(H_j)$ for $j = 1, 2, \cdots, p$.

Proof. From the relation $H_j \subset B_j \subset C_j + \Gamma_j$ it follows that $x \in H_j$ implies $|x - x^j| \leqq 2^{-1} \operatorname{diam} C_j = \operatorname{diam} G_j < \delta$; hence

$$|f(x) - f(x^j)| \leqq \eta \quad \text{for} \quad x \in H_j.$$

Since $\Phi(B_j) < \infty$ by Part 1, we know from Theorem 7.3 that

$$\int \nu(A_j x) d\Phi x = 0,$$

and conclude

$$\left|\int f(x)\nu(A_j, x)d\Phi x\right| = \left|\int [f(x) - f(x^j)]\nu(A_j, x)d\Phi x + f(x^j)\int \nu(A_j, x)d\Phi x\right|$$

$$\leqq \int_{H_j} |f(x) - f(x^j)|\, d\Phi x \leqq \eta \Phi(H_j).$$

Part 5. $\int f(x)\nu(A_0, x)d\Phi x = 0$.

Proof. Let

$$M = \sup_{x \in (A+B)} |f(x)|,$$

$$b = \inf_{x \in A_0, z \in B} |x - z|.$$

Since f is continuous on the compact set $A+B$, we have $M < \infty$. On the other hand

$$B(\text{closure } A_0) \subset (\sum_{j=1}^{p} C_j)(E_2 - \sum_{j=1}^{p} C_j) = 0,$$

the compact sets B and (closure A_0) are a positive distance apart, and $b > 0$. Hence 8.2 implies

$$|f'(x)| \leqq 8M/\pi b \quad \text{for} \quad x \in A_0.$$

Let f_1 and f_2 be such functions that

$$f(x) = (f_1(x), f_2(x)) \quad \text{for every } x,$$

and recall the Cauchy-Riemann equation

$$(D_1 f_1(x), D_1 f_2(x)) = f'(x) = (D_2 f_2(x), -D_2 f_1(x))$$

for $x \in A$.

Consequently

$$|D_j f_k(x)| \leqq |f'(x)| \leqq 8M/\pi b \quad \text{for} \quad x \in A_0;\ j = 1, 2;\ k = 1, 2.$$

It follows that

$$\{f_1\} + \{f_2\} \subset \Omega_1(A_0 + B_0)\Omega_2(A_0 + B_0).$$

Using Part 1, the last relation, and Theorem 7.3, we compute:

$$\begin{aligned}\int f(x)\nu(A_0, x)d\Phi x &= \int [f_1(x)\nu_1(A_0, x) - f_2(x)\nu_2(A_0, x)]d\Phi x \\ &\quad + i\int [f_1(x)\nu_2(A_0, x) + f_2(x)\nu_1(A_0, x)]d\Phi x \\ &= \int_{A_0} [D_1 f_1(x) - D_2 f_2(x)]dx \\ &\quad + i\int_{A_0} [D_2 f_1(x) + D_1 f_2(x)]dx = 0.\end{aligned}$$

Part 6. $|\int f(x)\nu(A, x)d\Phi x| \leqq \epsilon$.

Proof. The sets $A_0, A_1, A_2, \cdots, A_p$ are disjoint open sets whose boundaries have finite Φ measure and for which

$$\sum_{j=0}^{p} A_j \subset A \subset \text{closure} \sum_{j=0}^{p} A_j.$$

Hence we use 7.6, 6.8, and Parts 5, 4, 3 to conclude:

$$\left| \int f(x)\nu(A, x) d\Phi x \right| = \left| \sum_{j=0}^{p} \int f(x)\nu(A_j, x) d\Phi x \right|$$

$$\leqq \sum_{j=0}^{p} \left| \int f(x)\nu(A_j, x) d\Phi x \right| \leqq \sum_{j=1}^{p} \eta\Phi(H_j) \leqq \epsilon.$$

8.4 *Remark.* The hypotheses of 8.3 imply

$$f(z) = \frac{1}{2\pi} \int \frac{f(x)\nu(A, x)}{x - z} d\Phi x \quad \text{for} \quad z \in A.$$

This Cauchy formula is proved by the standard method, except that *cross-cuts are unnecessary* (here, and in similar situations), because Theorem 8.3 applies directly to the open set obtained from A by removing a closed circular disc with center z.

8.5 *Remark.* If p is a *Lipschitzian* function of the type described in 7.7, and if the hypotheses of 8.3 are satisfied, then

$$i \int f(x)\nu(A, x) d\Phi x = \mp \int_0^1 f[p(t)] i\tau p(t) \mid p'(t) \mid dt$$

$$= \pm \int_0^1 f[p(t)] \frac{p'(t)}{\mid p'(t) \mid} \mid p'(t) \mid dt$$

$$= \pm \int_0^1 f[p(t)] p'(t) dt$$

$$= \pm \int_0^1 f[p(t)] d_t p(t),$$

the sign depending on the sense in which p parametrizes B.

Hence Theorem 8.3 includes the well known *strong* form of *Cauchy's Theorem* for a simple closed curve.

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